COMMENT

General Relativistic Rossby-Haurwitz waves of a slowly and differentially rotating fluid shell

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Abstract. We show that, at first order in the angular velocity, the general relativistic description of Rossby-Haurwitz waves (the analogues of r waves on a thin shell) can be obtained from the corresponding Newtonian one after a coordinate transformation. As an application, we show that the results recently obtained by Rezzolla and Yoshida (2001) in the analysis of Newtonian Rossby-Haurwitz waves of a slowly and differentially rotating, fluid shell apply also in General Relativity, at first order in the angular velocity.

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1. Introduction

The investigation of r modes in rotating stars has been the subject of a widespread interest in the recent past. Among the numerous aspects considered so far (see Andersson and Kokkotas 2001, as well as Friedman and Lockitch 2001 for recent reviews), a particularly intriguing one has focussed on the effects induced by the differential rotation of the background stellar model (Spruit 1999, Rezzolla *et al* 2000, 2001a, 2001b) and whether this could prevent the excitation of the r modes (Karino *et al* 2001, Lindblom *et al* 2001). To alleviate the complications introduced by differential rotation in the r-mode eigenvalue problem, Rezzolla and Yoshida (2001) have recently investigated the properties of the analogues of r waves, the Rossby-Haurwitz waves (Haurwitz, 1940), on a differentially rotating, Newtonian thin shell of incompressible fluid. In this framework, the eigenvalue problem is much simpler to solve, but incorporates many of the mathematical properties of the corresponding eigenvalue problems for multidimensional Newtonian stars or for slowly-rotating relativistic stars. The most important of these properties is the possible existence of a singular behaviour for the eigenvalue problem. The purpose of this Comment is to extend, in a simplified model, the results of Rezzolla and Yoshida (2001) to the general relativistic case.

2. A useful coordinate transformation: the uniform rotation case

We consider here a slowly rotating body and make our calculations including terms up to first order in the angular velocity. To this order, the object remains spherical and Rossby-Haurwitz waves propagate in the spherical shell representing its surface. If the body is uniformly rotating and a spherical coordinate system (t, r, θ, ϕ) is used, the spacetime metric is given by (Hartle and Thorne 1968)

$$ds^2 = e^{2\nu(r)}dt^2 - R^2 d\theta^2 - R^2 \sin^2\theta \left[d\phi - \omega(r)dt\right]^2 \ , \tag{1}$$

and on the surface $r = R$, assumes the form

$$ds^2 = e^{2\nu_{\rm R}}dt^2 - R^2 d\theta^2 - R^2 \sin^2\theta \left[d\phi - \omega_{\rm R} dt\right]^2 \ , \tag{2}$$

where, the redshift $\nu_{\rm R} \equiv \nu(R)$ and the frame dragging angular velocity $\omega_{\rm R} \equiv \omega(R)$ are constants. Consider now the simple coordinate transformation

$$t \ \longrightarrow \ \widetilde{t} = e^{\nu_{\rm R}} t \ , \tag{3}$$

$$\phi \ \longrightarrow \ \widetilde{\phi} = \phi - \omega_{\rm R} t \ , \tag{4}$$

$$\theta \ \longrightarrow \ \widetilde{\theta} = \theta \ , \tag{5}$$

which brings the metric (2) into the evidently Minkowski form

$$ds^2 = d\widetilde{t}^{\,2} - R^2 (d\widetilde{\theta}^2 + \sin^2\widetilde{\theta} d\widetilde{\phi}^2) \ . \tag{6}$$

The coordinate transformation (3)–(5) shows that, at first order in Ω and limited to the spherical surface $r = R$, the general relativistic description of the physics on the slowly and uniformly rotating thin shell can be done entirely in Newtonian terms. We will exploit this feature to deduce, from known Newtonian results, the properties of general relativistic Rossby-Haurwitz waves in a slowly, differentially rotating fluid shell. Hereafter, we will indicate with a tilde all of the Newtonian quantities.

As a first application of this procedure, we recall that for a shell rotating with a uniform angular angular velocity $\widetilde{\Omega}$, the Newtonian dispersion relation for the Rossby-Haurwitz modes at first order in $\widetilde{\Omega}$ takes the form (Haurwitz 1940; Stewartson and Rickard 1969)

$$\widetilde{\Pi} = \widetilde{\Omega} - \frac{2\widetilde{\Omega}}{l(l+1)} \ . \tag{7}$$

Here $\widetilde{\Pi} \equiv \widetilde{\sigma}/m$ is the phase velocity‡ and $\widetilde{\sigma}$ is the eigenfrequency of the mode with wavenumbers m and l. Using the coordinate transformation (3)–(5) we can write

$$\widetilde{\Omega} \equiv \frac{d\widetilde{\phi}}{d\widetilde{t}} = e^{-\nu_{\rm R}}(\Omega - \omega_{\rm R}) \ , \qquad\qquad \widetilde{\Pi} = e^{-\nu_{\rm R}}(\Pi - \omega_{\rm R}) \ . \tag{8}$$

Inserting now expressions (8) in (7) we immediately obtain

$$\Pi = \Omega - \frac{2(\Omega - \omega_{\rm R})}{l(l+1)} \ , \tag{9}$$

which represents the general relativistic expression for the Newtonian dispersion relation (7). As expected, our expression (9) agrees with the dispersion relation, evaluated on the surface $r = R$, and derived by Kojima (1997, 1998; see also Beyer and Kokkotas 1999; Lockitch *et al* 2001), through direct but lengthy relativistic calculations.

‡ In Rezzolla and Yoshida (2001) the phase velocity was indicated as $\omega_{\rm ph}$, but this symbol is not used here to avoid confusion with the frame dragging angular velocity ω.

3. The differential rotation case

Although the rotating body generating the line element (1) is uniformly rotating, we allow for a differential rotation $\widetilde{\Omega} = \widetilde{\Omega}(\mu)$, with $\mu \equiv \cos\theta$, be present on the thin shell§. Already for this simplified model, an analytic dispersion relation for the Rossby-Haurwitz waves propagating on the shell is not available, not even in the Newtonian limit. However, we can express the eigenvalue problem for the waves on a differentially rotating thin shell of radius $r = R$ through the ordinary differential equation (Rezzolla and Yoshida 2001),

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\widetilde{\chi}}{d\mu}\right] - \frac{m^2}{1-\mu^2}\widetilde{\chi} - \frac{2\widetilde{\Omega}}{\widetilde{\Pi}-\widetilde{\Omega}}\left[1 + \left(\frac{2\mu}{\widetilde{\Omega}}\right)\frac{d\widetilde{\Omega}}{d\mu} - \left(\frac{1-\mu^2}{2\widetilde{\Omega}}\right)\frac{d^2\widetilde{\Omega}}{d\mu^2}\right]\widetilde{\chi} = 0\ , \quad (10)$$

where $\widetilde{\chi}(\mu)/\sin\theta = u^\theta$ is the eigenfunction of the mode, with u^θ being the θ component of the velocity perturbation. Because of the coordinate transformation (3)–(5), the numerical solution of the general relativistic analogue of equation (10) is not necessary. Rather, once the (tilde) Newtonian solution of (10) is known, the properties of general relativistic Rossby-Haurwitz waves of a slowly and differentially rotating fluid shell at first order in the shell's angular velocity are simply determined as

$$\Omega(\mu) = e^{\nu_{\rm R}}\widetilde{\Omega}(\mu) + \omega_{\rm R}\ , \quad (11)$$

$$\sigma = e^{\nu_{\rm R}}\widetilde{\sigma} + m\omega_{\rm R}\ , \quad (12)$$

$$\chi = \widetilde{\chi}\ . \quad (13)$$

An important issue concerning the eigenvalue problem for Rossby-Haurwitz waves is that of "corotation", i.e. of whether differential rotation could bring the ratio $\widetilde{\Pi}/\widetilde{\Omega}(\mu)$ to be one. If this would happen, the denominator of the third term in equation (10) would vanish making the eigenvalue problem a singular one and preventing the existence of the modes.

In the case the shell is *uniformly rotating* at an angular frequency Ω, it is straightforward to show that in General Relativity and at first order in Ω, corotation cannot take place since in this case $\omega_{\rm R}/\Omega < 1$ and expression (9) gives

$$1 - \frac{\Pi}{\Omega} = \frac{2}{l(l+1)}\left(1 - \frac{\omega_{\rm R}}{\Omega}\right) > 0\ . \quad (14)$$

We next consider the shell to be *differentially rotating* and whether corotation can occur for a sufficiently large degree of differential rotation. For a number of different laws of differential rotation, Rezzolla and Yoshida (2001) have shown that the solution of equation (10) does not show evidence of corotation and that $\widetilde{\Pi}/\widetilde{\Omega}_{\rm E} < 1$ even for asymptotically large values of differential rotation, with $\widetilde{\Omega}_{\rm E} \equiv \widetilde{\Omega}(\mu = 0)$ being the minimum angular velocity. Using expressions (8) it is straightforward to deduce that this is true also for the general relativistic analogue of equation (10), for which

$$1 - \frac{\Pi}{\Omega_{\rm E}} = \frac{e^{\nu_{\rm R}}\widetilde{\Omega}_{\rm E}}{e^{\nu_{\rm R}}\widetilde{\Omega}_{\rm E} + \omega_{\rm R}}\left(1 - \frac{\widetilde{\Pi}}{\widetilde{\Omega}_{\rm E}}\right) > 0\ . \quad (15)$$

As a result, the general relativistic eigenvalue problem for Rossby-Haurwitz waves for the line element (1) does not show any singular behaviour.

While we regard this result as interesting and indicative, it does not necessarily imply that the eigenvalue problem for r waves cannot be singular for a uniformly, slowly rotating

§ If the underlying body is also rotating differentially the frame dragging angular velocity ω acquires a θ-dependence already at first order in the angular velocity. In this case ω needs to be expressed in a series expansion in terms of vector spherical harmonics and the line element (1) to be suitably corrected (Hartle 1970).

star (see Kojima 1998, but also Ruoff and Kokkotas 2001a, 2001b, and Yoshida 2001). This is because, by construction, the general relativistic equivalent of equation (10) cannot account for the radial dependence of the frame dragging angular velocity ω. This is at the origin of a singular behaviour reported at first order in the slow-rotation approximation, even for uniformly rotating stars, but which seems to disappear when terms of $\mathcal{O}(\Omega^2)$ are taken into account (Yoshida and Lee 2001). In other words, the singular eigenvalue problem due to differential rotation on a shell and the singular eigenvalue problem due, in the slow-rotation approximation, to the frame dragging angular velocity, are mathematically similar but physically distinct.

As a final remark we underline that the procedure followed here is generic and the coordinate transformation (3)–(5) can be used any time that first order general relativistic effects need to be considered on a thin, slowly rotating shell.

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